

The Pythagoras number and the u -invariant of Laurent series fields in several variables

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Abstract

Let k be a field of characteristic different from 2 and $F_n = k((t_1, \dots, t_n))$ a Laurent series field in $n \geq 2$ variables over k . We study the Pythagoras number $p(F_n)$ and the u -invariant $u(F_n)$ (in the sense of Elman–Lam) of the field F_n . We prove an equality which relates $p(F_2)$ (resp. $u(F_2)$) to the Pythagoras numbers (resp. the u -invariants) of the rational function fields $k'(t)$ for all finite extensions k' of k . This enables us to get the exact value of $p(k((t_1, t_2)))$ when k is a number field. In particular, we obtain $p(\mathbb{Q}((t_1, t_2))) = 5$. The formula for the u -invariant implies the finiteness of $u(k((t_1, t_2)))$ for any finitely generated extension k of \mathbb{R} . In general, we show that $p(F_n) \leq p(F_{n-1}(t))$ and $u(F_n) \leq u(F_{n-1}(t))$. This leads us to a proof of the equality $p(\mathbb{R}((t_1, t_2, t_3))) = 4$. We also make two conjectures for general n based on our specific results.

1 Introduction

Let K be a field of characteristic different from 2. The Pythagoras number $p(K)$ of K is the smallest integer $p \geq 1$ (or ∞) such that every sum of (finitely many) squares in K can be written as a sum of p squares, and the u -invariant $u(K)$ in the sense of Elman–Lam is the supremum of the dimensions of all anisotropic *torsion* quadratic forms over K .

In this paper we will mainly focus on the case of a Laurent series field $F_n = k((t_1, \dots, t_n))$ in $n \geq 2$ variables over a field k of characteristic $\neq 2$. (The $n = 1$ case is classical.) For the Pythagoras number, it turns out (cf. Prop. 3.3) that the case with k a nonreal field (i.e. a field in which -1 is a sum of squares) is much easier than the case with k a real field.

Consider the case with k real. As far as we know, almost all known results about the Pythagoras number $p(F_n)$ come from an influential paper of Choi, Dai, Lam and Reznick [CDLR82], published in Crelle’s journal in the 1980’s. First, in the case $n = 2$, they prove that $p(F_2) = 2$ if k is a hereditarily pythagorean field (e.g. $k = \mathbb{R}$ the field of real numbers), by showing that the ring of formal power series $k[[t_1, t_2]]$ has Pythagoras number 2. For a general base field k , they show that if the Pythagoras number of the rational function field $k(t)$ is bounded by a 2-power, then $p(F_2)$ is bounded by the same

2-power. Also established in that paper is the fact that the ring $k[[t_1, \dots, t_n]]$ has an infinite Pythagoras number when $n \geq 3$. (The Pythagoras number of a ring can be defined as in the case of fields.) They were then led to ask the question about the finiteness and the precise value of the Pythagoras number of $k((t_1, \dots, t_n))$ when $n \geq 3$ and k is, for instance, the field \mathbb{R} (cf. [CDLR82, p.70 and p.80, §9, Problem 6]).

The study of the u -invariant $u(F_n)$ was also pioneered by the paper [CDLR82]. There it is proved that $u(F_2) = 4$ when k is an algebraically closed field. In [CTOP02], this is extended to any finite extension of F_2 for a real closed field k . Some more recent results in the nonreal case can be found in [HHK09], [HHK11], [Lee10] and [Hu11]. All these known results are concerned with the 2-dimensional case, i.e., finite extensions of F_2 , and until now no specific results about $u(F_2)$ seem to have been noticed for a general real field k .

In this paper we take into account some newly developed methods and consider at the same time the Laurent series fields $F_n = k((t_1, \dots, t_n))$ and the rational function fields $F_n(t)$ over them. We try to relate their Pythagoras numbers and u -invariants to those of algebraic function fields over the base field k . Our approach relies on some extensions of the valuation-theoretic arguments used by Becher, Grimm and Van Geel in [BGVG12]. We also use some results in that paper, which in turn, build upon a patching method developed by Harbater, Hartmann and Krashen ([HHK09]) and a local-global principle ([CTPS12, Thm. 3.1]) proved with that method by Colliot-Thélène, Parimala and Suresh.

First present our main results about the Pythagoras number.

In the $n = 2$ case, we prove that $p(F_2)$ is equal to the supremum of the Pythagoras numbers of the rational function field $k'(t)$ over finite extensions k' of k (cf. Thm. 4.2). As an application of this formula, we show that the equality $p(k((t_1, t_2))) = p(k(t))$ holds for any number field k (cf. Thm. 4.4). In particular, we get $p(\mathbb{Q}((t_1, t_2))) = 5$. To our knowledge, if k is real and $p(k(t))$ is not a 2-power, no result on the precise value of $p(k((t_1, t_2)))$ has been obtained previously.

Also, we have been able to show that the rings

$$k[[x, t]], \quad k((x, t)), \quad k[x][[t]] \quad \text{and the fraction field of } k[x][[t]]$$

all have the same Pythagoras number as $k((t))(x)$.

For Laurent series in three or more variables, by refining and generalizing some Weierstrass-type arguments in [CDLR82] we obtain the general relation $p(F_n) \leq p(F_{n-1}(t))$. This enables us to prove (in Thm. 6.1) that the field $k((t_1, t_2, t_3))$ has Pythagoras number 4 when k is a real closed field. Previously, even the finiteness of this Pythagoras number was unknown.

For general $n \geq 2$, we conjecture that the Pythagoras numbers $p(F_n)$ and $p(F_{n-1}(t))$ ought to be determined by the Pythagoras numbers of rational function fields in $n - 1$ variables over finite extensions of k (cf. Conjecture 7.2). This conjecture, if true, will certainly yield some finiteness results on $p(F_n)$ (and $p(F_{n-1}(t))$) in a couple of sample cases, e.g., when $k = \mathbb{R}$ or $k = \mathbb{Q}$.

In the case $n = 2$, Karim Becher told us his guess that $p(k((t_1, t_2))) = p(k(t))$ and our results do provide some more evidence for this equality to hold. In fact, our results imply that this equality is equivalent to two (equivalent) conjectures of Becher, Grimm and Van Geel ([BGVG12, Conjectures 4.9 and 4.10]). If their conjectures hold, our conjecture will mean that $p(F_n)$ and $p(F_{n-1}(t))$ are both equal to $p(k(t_1, \dots, t_{n-1}))$.

For the u -invariant we have obtained similar results. For any (real or nonreal) field k , the u -invariants of the fields $k((x, t))$, $k((t))(x)$ and the fraction field of $k[x][[t]]$ are shown to be the same and equal to twice the supremum of the u -invariants $u(\ell(x))$ for all finite extensions ℓ/k (cf. Thm. 4.9). As a consequence of this theorem, the finiteness of $u(k((x, t)))$ is proved when k is a finitely generated extension of a real closed field. We conjecture that this theorem generalizes to Laurent series in more variables as well (cf. Conjecture 7.3).

Notation and Convention. Throughout what follows, we will assume 2 is invertible in all rings (e.g. fields) under consideration.

Letters t, t_1, \dots and x, y, x_1, \dots will denote independent variables. If k is a field, then $k[[t_1, \dots, t_n]]$ denotes the ring of formal power series in the variables t_1, \dots, t_n over k , its fraction field $k((t_1, \dots, t_n))$ is the corresponding field of Laurent series and $k(x_1, \dots, x_m)$ denotes a rational function field in m variables over k .

By an *algebraic function field in $d \geq 0$ variables* over a field k we shall mean a finitely generated field extension L/k of transcendence degree d .

A discrete valuation will always be assumed normalized (nontrivial) and of rank 1. If v is a discrete valuation on a field F , we often denote by $\kappa(v)$ the residue field of v and by F_v the completion of F with respect to v .

For an integral domain A , $\text{Frac}(A)$ denotes its field of fractions.

For quadratic forms we follow standard notation as used in [Lam05]. In particular, for a field F , $W(F)$ denotes the Witt group of quadratic forms over F . A quadratic form over F is called *torsion* if its class in $W(F)$ is a torsion element.

2 Some field invariants related to quadratic forms

(2.1) For a field K (of characteristic $\neq 2$), we denote

$$D_K(n) := \{x_1^2 + \dots + x_n^2 \mid x_i \in K\} \quad \text{for each } n \geq 1,$$

and

$$D_K(\infty) := \bigcup_{n \geq 1} D_K(n).$$

The **level** $s(K)$ and the **Pythagoras number** of the field K are defined by

$$s(K) := \inf\{n \geq 1 \mid -1 \in D_K(n)\} \in \mathbb{N} \cup \{\infty\},$$

and

$$p(K) := \inf\{n \geq 1 \mid D_K(n) = D_K(\infty)\} \in \mathbb{N} \cup \{\infty\}.$$

By a theorem of Pfister, $s(K) \in \{2^i \mid i \geq 0\} \cup \{\infty\}$.

The field K is called (**formally**) **real** (resp. **nonreal**) if $s(K) = \infty$ (resp. $s(K) < \infty$); **pythagorean** if $p(K) = 1$. If K is real and every finite real extension L of K is pythagorean, then we say K is **hereditarily pythagorean**. A real field K is called **real closed** if no nontrivial algebraic extension of K is real.

The **u -invariant** of K in the sense of Elman–Lam ([EL73]) is defined as

$$u(K) = \sup\{\dim \phi \mid \phi \text{ an anisotropic } \textit{torsion} \text{ quadratic form over } K\}.$$

For a nonreal field, this agrees with Kaplansky’s definition. For a real field K , $u(K)$ must be even or ∞ . Unless the field K is real and pythagorean (\iff the Witt group $W(K)$ is torsion free $\iff u(K) = 0$), one has $1 \leq p(K) \leq u(K)$.

(2.2) For a commutative ring A (in which 2 is invertible), one can define similarly the sets $D_A(n)$, $D_A(\infty)$ and the invariants $s(A)$, $p(A)$.

Some basic facts:

(1) Whenever $s(A) < \infty$, one has $s(A) \leq p(A) \leq s(A) + 1$.

(2) If B is a quotient or a localization of A , one has $p(B) \leq p(A)$.

If $A \rightarrow C$ is a ring homomorphism, then $s(C) \leq s(A)$.

Some special cases:

(i) If A is an integral domain with fraction field K , then $s(K) \leq s(A)$ and $p(K) \leq p(A)$.

(ii) If A is a local ring with residue field k , then $s(k) \leq s(A)$ and $p(k) \leq p(A)$. If moreover A is henselian, then $s(A) = s(k)$ and $A^* \cap D_A(\infty) \subseteq A^* \cap D_A(p(k))$.

(3) By the Cassels–Pfister theorem ([Lam05, Thm. IX.1.3]), if k is a field, one has $p(k[t]) = p(k(t))$ and $s(k[t]) = s(k(t))$.

(4) (Colliot-Thélène–Kneser, cf. [CLRR80, Thm. 4.5]) Let K be the fraction field of a valuation ring A with residue field k . Then for any regular quadratic form ϕ over A , ϕ represents an element $a \in A$ over A if and only if ϕ represents a over K . It follows in particular that $p(k) \leq p(A) = p(K)$ and $s(k) \leq s(A) = s(K)$.

(5) Combining (2)–(4) we see that if k is a field, then

$$s(k(t)) = s(k[t]) = s(k) = s(k[[t]]) = s(k((t)))$$

and

$$p(k) \leq p(k[t]) = p(k(t)), \quad p(k) \leq p(k[[t]]) = p(k((t))).$$

Further, by [Lam05, Thm. IX.2.1], one has for any $a \in k^*$ and $n \geq 1$,

$$a \in D_k(n) \iff t^2 + a \in D_{k(t)}(n+1).$$

One thus obtains

$$p(k(t)) \begin{cases} \geq p(k) + 1 & \text{if } k \text{ is real,} \\ = s(k) + 1 & \text{if } k \text{ is nonreal.} \end{cases}$$

(In the nonreal case, one can deduce $p(k(t)) \leq s(k) + 1$ by using (1) together with the inequality $s(k(t)) \leq s(k)$, which follows from (2).)

(2.3) Let K be a field. As in [BGVG12], to avoid case distinction in some statements we set

$$p'(K) = \begin{cases} p(K) & \text{if } K \text{ is real} \\ s(K) + 1 & \text{if } K \text{ is nonreal.} \end{cases}$$

As noted in (2.2), the following inequalities always hold:

$$p(K) \leq p'(K) \leq p(K) + 1.$$

Note also that $p(K(t)) = p'(K(t))$ for any (real or nonreal) field K .

3 Results from valuation theory

In this section, we briefly review and further develop some of the valuation-theoretic arguments used in [BGVG12]. We shall apply this method to get some lower bounds for the Pythagoras number and the u -invariant of Laurent series fields.

Let K be a field. A discrete valuation v of K is called *nondyadic* if the residue field $\kappa(v)$ of v has characteristic $\neq 2$.

Proposition 3.1 ([BGVG12, Propositions 4.5 and 5.2], [Sch09, Prop. 5]). *Let v be a nondyadic discrete valuation of a field K . Then*

$$p'(K) \geq p(K) \geq p'(\kappa(v)) \quad \text{and} \quad u(K) \geq 2u(\kappa(v)).$$

The equalities hold if v is henselian (meaning that the discrete valuation ring associated to v is henselian).

(3.2) Prop. 3.1 generalizes the classical facts

$$p'(K((t))) = p(K((t))) = p'(K) \quad \text{and} \quad u(K((t))) = 2u(K)$$

for any field K (cf. [Lam05, p.398, Examples XI.5.9 (6)] and [Pfi95, p.114, Examples 1.7 (2)]).

Also, if L/K is a finite separable field extension, then there is a discrete valuation v on $K(t)$ whose residue field $\kappa(v)$ is isomorphic to L . So, by Prop. 3.1, $p(K(t)) \geq p'(L)$ and $u(K(t)) \geq 2u(L)$. If M/L is a purely inseparable extension of fields of characteristic $p > 2$, then a theorem of Springer (cf. [Lam05, p.194, Thm. VII.2.7]) implies that $u(L) \leq u(M)$ and $p'(M) = s(M) + 1 = s(L) + 1 = p'(L)$. On the other hand, for every element $a \in M$, one has $a^r \in L$ for some odd integer $r \geq 1$ (e.g., $r = p^e$ for some large integer $e \geq 1$). From this it follows that $u(M) = u(L)$. Hence, we have

$$p(K(t)) \geq \sup\{p'(L) \mid L/K \text{ a finite field extension}\}$$

and

$$u(K(t)) \geq 2 \sup\{u(L) \mid L/K \text{ a finite field extension}\}.$$

As of now, k denotes a field (of characteristic $\neq 2$), $R_n = k[[t_1, \dots, t_n]]$ denotes the ring of power series in $n \geq 1$ variables t_1, \dots, t_n over k and $F_n = k((t_1, \dots, t_n))$ is the fraction field of R_n , i.e., the field of Laurent series in the variables t_1, \dots, t_n . Sometimes we use the convention $R_0 = F_0 = k$.

Proposition 3.3. *If k is nonreal, then for every $n \geq 1$ one has*

$$s(F_n) = s(k) \quad \text{and} \quad p(F_n) = s(k) + 1 = p(F_n(t)).$$

Proof. Since $p(K(t)) = s(K) + 1$ for any nonreal field K (cf. (2.2) (5)), we need only prove the equalities $s(F_n) = s(k)$ and $p(F_n) = s(k) + 1$.

We use induction on n , the case $n = 1$ being discussed in (2.2) and (3.2).

If $n \geq 2$, the inclusions $k \subseteq F_n \subseteq F_{n-1}((t_n))$ yield inequalities

$$s(F_{n-1}((t_n))) \leq s(F_n) \leq s(k).$$

But $s(F_{n-1}((t_n))) = s(F_{n-1}) = s(k)$ by the $n = 1$ case and the induction hypothesis. This proves $s(F_n) = s(k)$.

Now for the Pythagoras number, we have $p(F_n) \leq s(F_n) + 1 = s(k) + 1$. On the other hand, putting $s = s(k)$, the form $(s + 1)\langle 1 \rangle$ is isotropic over k hence also over F_n . So we have in particular $t_n \in D_{F_n}(s + 1)$. We claim that $t_n \notin D_{F_n}(s)$, so that $p(F_n) \geq s + 1$ as desired. It is enough to show that $t_n \notin D_{F_{n-1}((t_n))}(s)$. Indeed, if t were represented by the form $s.\langle 1 \rangle$ over $F_{n-1}((t_n))$, then it would be represented by $s.\langle 1 \rangle$ over $F_{n-1}[[t_n]]$ (by (2.2) (4)), say

$$t_n = \alpha_1^2 + \cdots + \alpha_s^2 \quad \text{with each } \alpha_i \in F_{n-1}[[t_n]].$$

Here not all of the α_i can be divisible by t_n , so modulo t_n this shows that the form $s.\langle 1 \rangle$ is isotropic over F_{n-1} . But this leads to a contradiction since $s(F_{n-1}) = s$. \square

Here is another result of interest to us.

Theorem 3.4 ([BGVG12, Theorems 4.8, 6.4 and Coro. 6.9]).

(i) $p(k(t)) \leq p(k((t))(x))$ and these two Pythagoras numbers are bounded by the same 2-power.

(ii) $p(k((t))(x)) = \sup\{p(\ell(x)) \mid \ell/k \text{ a finite field extension}\}.$

(iii) $u(k((t))(x)) = 2 \sup\{u(\ell(x)) \mid \ell/k \text{ a finite field extension}\}.$

Proof. (i) The first assertion is contained in [Sch01, Prop. 5.17]. The proof for the second assertion already appeared in [CDLR82, Thm. 5.18]. (See also [BGVG12, Thm. 4.8]).

(ii) [BGVG12, Coro. 6.9].

(iii) [BGVG12, Thm. 6.4]. \square

Lemma 3.5. *Let $m \geq 0$ be an integer.*

(i) *For each $1 \leq i \leq n$, one has*

$$p(F_n(x_1, \dots, x_m)) \geq p(F_{n-i}(x_1, \dots, x_{m+i-1})).$$

(ii) *Assume $n \geq 2$ and let ℓ/k be a finite field extension. Then*

$$p(F_n(x_1, \dots, x_m)) \geq p(\ell(t_1, \dots, t_{n-1}, x_1, \dots, x_m)).$$

In particular,

$$p(F_n) \geq p(\ell(t_1, \dots, t_{n-1})) \quad \text{and} \quad p(F_n(t)) \geq p(\ell(t_1, \dots, t_n)).$$

Proof. The strategy is to find a suitable discrete valuation and to apply Prop. 3.1.

(i) Consider the regular affine scheme

$$X = \operatorname{Spec}(R_n[x_1, \dots, x_m]) = \operatorname{Spec}(k[[t_1, \dots, t_n]][x_1, \dots, x_m])$$

and its closed subscheme Y defined by $t_{n-i+1} = \dots = t_n = 0$. The blowup $\operatorname{Bl}_Y X$ of X along Y contains an exceptional divisor E that is isomorphic to \mathbb{P}_Y^{i-1} . The field

$$F_{n-i}(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+i-1})$$

is isomorphic to the function field of E , so it is the residue field of a discrete valuation of $F_n(x_1, \dots, x_m)$, which is the function field of X .

(ii) First assume $m = 0$. We want to realize $\ell(x_1, \dots, x_{n-1})$ as the residue field of a discrete valuation of F_n . Let $X = \operatorname{Spec}(R_n)$ and let $X' \rightarrow X$ be the blowup of X at its closed point P . The exceptional divisor E in X' is isomorphic to \mathbb{P}_k^{n-1} . Since $n \geq 2$, there is a closed point $Q \in E$ whose residue field $\kappa(Q)$ is isomorphic to ℓ . Blowing up X' at the point Q we get an exceptional divisor E' which is isomorphic to \mathbb{P}_ℓ^{n-1} . Now the generic point of E' defines a discrete valuation of F_n whose residue field is $\ell(x_1, \dots, x_{n-1})$.

For general m the argument is similar: Let X, P, Q and so on be as above. Then one just needs to consider the blowups $\operatorname{Bl}_{P \times \mathbb{A}^m}(X \times \mathbb{A}^m)$ and $\operatorname{Bl}_{Q \times \mathbb{A}^m}(X' \times \mathbb{A}^m)$. \square

One can also apply the ideas in the above proof to get similar results for the u -invariant. In particular, we have:

Lemma 3.6. *For any $n \geq 2$, one has*

$$u(F_n) \geq 2 \sup\{u(\ell(t_1, \dots, t_{n-1})) \mid \ell/k \text{ a finite field extension}\}.$$

Remark 3.7. The essential ideas in the proof of Lemma 3.5 can be applied more generally to the fraction field K of any regular local ring A . If k is the residue field of A and A has Krull dimension $n \geq 2$, one has for example

$$p(K) \geq \sup\{p'(\ell(t_1, \dots, t_{n-1})) \mid \ell/k \text{ a finite field extension}\}.$$

In the 2-dimensional case a weaker version of this estimate was given in [Sch01, Lemma 5.16].

4 Laurent series in two variables

Lemma 4.1 ([CDLR82, Thm. 5.20]). *Let k be a real field. Consider the rings*

$$B := k[[t]][x] \subseteq A := k[x][[t]] \subseteq R := k[[x, t]].$$

Then for every $m \geq 1$, $D_A(m) = A^2 \cdot D_B(m)$ and $D_R(m) = R^2 \cdot D_B(m)$, namely, every sum of m squares in A is of the form $a^2 \cdot b$, where $a \in A$ and b is a sum of m squares in B , and similarly for R and B .

Proof. Here we give the proof for the relation $D_A(m) = A^2 \cdot D_B(m)$ because we feel that in the original proof of this fact in [CDLR82] some points have to be further clarified.

Consider $f = \sum_{i=1}^m g_i^2$ with $g_i \in A = k[x][[t]]$. If $t \mid f$, then $t \mid g_i$ for each i (since $s(k(x)) = \infty$). Canceling t^2 if necessary, we may assume that $t \nmid f$. Writing

$$f = \sum_{j=0}^{\infty} f_j(x)t^j \quad \text{with each } f_i \in k[x],$$

we have $f_0 \neq 0$ in $k[x]$.

We claim that for every $\phi \in A = k[x][[t]]$, there is an expression

$$\phi = qf + r$$

with

$$q = \sum_{j=0}^{\infty} q_j(x)t^j \in A = k[x][[t]], \quad r = \sum_{j=0}^{\infty} r_j(x)t^j \in B = k[[t]][x] \subseteq A$$

and $\deg_x r_0(x) < \deg_x f_0(x)$.

Assuming the claim, we can write $g_i = h_i f + r_i$ where $h_i \in A$ and $r_i \in B \subseteq A$ with $\deg_x r_i(x, 0) < \deg_x f(x, 0)$. Now the relation $f(x, 0) = \sum g_i(x, 0)^2$ implies $2 \deg_x g_i(x, 0) \leq \deg_x f(x, 0)$ for each i (since $s(k) = \infty$). Thus, from the equation $g_i(x, 0) = h_i(x, 0)f(x, 0) + r_i(x, 0)$ we see that $h_i(x, 0) = 0 \in k[x]$, i.e., $t \mid h_i$ in A . Then we get

$$\sum r_i^2 = \sum (g_i - h_i f)^2 = \sum g_i^2 - 2f \sum g_i h_i + f^2 \sum h_i^2 = fu,$$

where

$$u := 1 - 2 \sum g_i h_i + f \sum h_i^2$$

Since $t \mid h_i$ in A for every i , u is a unit and is a square in A . This shows $f \in A^2 \cdot D_B(m)$ as desired.

It remains to prove our claim.* Write $\phi = \sum_{j \geq 0} \phi_j t^j$. We need only to construct inductively two sequences $\{q_j\}_{j \geq 0}$, $\{r_j\}_{j \geq 0}$ of elements in $k[x]$ with each $\deg_x r_j < \deg_x f_0$ such that

$$\phi_N = \sum_{j=0}^N q_j f_{N-j} + r_N, \quad \forall N \geq 0.$$

For $N = 0$, it suffices to apply the Euclidean division algorithm to get $\phi_0 = q_0 f_0 + r_0$. Suppose we have constructed q_j and r_j for $j \leq N$. Then q_{N+1} and r_{N+1} can be determined by the equation

$$\phi_{N+1} - \sum_{j=0}^N q_j f_{N+1-j} = q_{N+1} f_0 + r_{N+1},$$

which is again obtained from the Euclidean division algorithm. This completes the proof of the lemma. \square

*Here, if we use the Weierstrass division theorem as in [CDLR82, p.70], we feel that we can only get an expression with $q \in k[x, t]$, and there seems to be no obvious reason why q has to lie in A .

The following result strengthens [CDLR82, Coro. 5.21].

Theorem 4.2. *Let k be a real field. Then the rings*

$$k[[x, t]], k[x][[t]], k[[t]][x], k((x, t)), \text{Frac}(k[x][[t]]) \text{ and } k((t))(x)$$

have the same Pythagoras number, which is equal to

$$\sup\{p(\ell(x)) \mid \ell/k \text{ a finite field extension}\}.$$

Proof. From Lemma 4.1 it follows that

$$p(k((x, t))) \leq p(k[[x, t]]) \leq p(k[x][[t]]) \leq p(k[[t]][x])$$

and

$$p(k((x, t))) \leq \text{Frac}(k[x][[t]]) \leq p(k((t))(x)).$$

On the other hand, the proof of [CDLR82, Thm. 5.18] has actually shown the equality $p(k[[t]][x]) = p(k((t))(x))$. So it suffices to show

$$p(k((x, t))) = p(k((t))(x)) = \sup\{p(\ell(x)) \mid \ell/k \text{ a finite field extension}\}.$$

But this follows by combining Thm. 3.4 (ii) and Lemma 3.5 (ii). \square

Example 4.3. Let k a real field and $F = k((t_1, t_2))$ a field of Laurent series in two variables over k . From Thm. 3.4 (i) and Thm. 4.2, we see that $p(k(t)) \leq p(F)$ and that the two Pythagoras numbers $p(k(t))$ and $p(F)$ are bounded by the same 2-power. Here are some applications of these estimates.

The first two examples below should be compared with [CDLR82, Coro. 5.22].

(1) Let k_0 be a real closed field (e.g., $k_0 = \mathbb{R}$) and let k be a real algebraic function field in $d \geq 0$ variables over k_0 . Then $d + 2 \leq p(F) \leq 2^{d+1}$.

Indeed, it is well known that $p(k(t)) \leq 2^{d+1}$ (cf. [Lam05, p.397, Examples XI.5.9 (4)]), so we have $p(F) \leq 2^{d+1}$. That $p(k(t)) \geq d + 2$ follows from a recent theorem of David Grimm ([Gri12, Thm. 1.1]).

The case with $d = 0$ yields $p(k_0((t_1, t_2))) = 2$ ([CDLR82, Coro. 5.14]). If $d = 1$, we get $3 \leq p(k(t)) \leq p(F) \leq 4$.

If $d \geq 1$ and $k = k_0(x_1, \dots, x_d)$ is a rational function field over k_0 , then

$$2^{d+1} \geq p(F) \geq p(k(t)) \geq p(k_0(t, x_1)) + d - 1 = d + 3.$$

In particular, we have $p(k_0(x)((t_1, t_2))) = 4$.

(2) Let k be a real algebraic function field in $d \geq 0$ variables over \mathbb{Q} .

If $d = 0$, i.e., k is a real number field, then $4 \leq p(k(t)) \leq 5$ (cf. [Pfi95, Chap. 7, Thm. 1.9]). In this case we have in fact $p(F) = p(k(t))$ (see Thm. 4.4 below).

Next assume $d \geq 1$. Thanks to Jannsen's work on higher class field theory ([Jan09, Coro. 0.2]) and the proof of Milnor's conjecture by Voevodsky and his coauthors ([Voe03] and [OVV07]), we can deduce $p(k(t)) \leq 2^{d+2}$ from [CTJ91, Thm. 4.1]. On the other

hand, there is a real number field k_0 such that $k_0(t, x_1, \dots, x_{d-1})$ is the residue field of a discrete valuation of $k(t)$ [Gri12, Prop. 2.2]. Hence

$$p(k(t)) \geq p(k_0(t, x_1, \dots, x_{d-1})) \geq p(k_0(t)) + d - 1 \geq d + 3.$$

We thus obtain

$$d + 3 \leq p(k(t)) \leq p(F) \leq 2^{d+2} \quad \text{when } d \geq 1.$$

If k is a rational function field, we can give a better lower bound. For example, if $k = \mathbb{Q}(x_1, \dots, x_d)$, we have $p(k(t)) \geq d + p(\mathbb{Q}(t)) = d + 5$. In particular,

$$6 \leq p(\mathbb{Q}(x)((t_1, t_2))) \leq 8.$$

(3) If $k = k_0((x, y))$ is a Laurent series field in two variables over a real closed field k_0 , we will show in Thm. 6.1 that $p(k(t)) = 4$. So we get $p(F) = 4$ in this case. In particular,

$$p(\mathbb{R}((x, y))((t_1, t_2))) = 4.$$

Theorem 4.4. *If k is a number field, then $p(k((x, t))) = p(k(t))$.*

Proof. The nonreal case was treated in Prop. 3.3. In the real case we may apply Thm. 4.2. If $p(k(t)) = 4$, we get $p(k((x, t))) = 4$ as in Example 4.3. Otherwise, $p(k(t)) = 5$ and $p(\ell(t)) \leq 5$ for all finite extensions ℓ/k (cf. [Pfi95, Chap. 7, Thm. 1.9]). \square

Corollary 4.5. *All the following rings have Pythagoras number equal to 5:*

$$\mathbb{Q}[x, t], \mathbb{Q}[x][t], \mathbb{Q}[t][x], \mathbb{Q}((x, t)), \text{Frac}(\mathbb{Q}[x][t]) \text{ and } \mathbb{Q}((t))(x).$$

Previously, the Pythagoras numbers of $\mathbb{Q}[x, t]$, $\mathbb{Q}[x][t]$, $\mathbb{Q}((x, t))$ and $\text{Frac}(\mathbb{Q}[x][t])$ were only known to be in the interval $[5, 8]$ (cf. [CDLR82, p.74]).

Now recall the following conjecture of Becher, Grimm and Van Geel.

Conjecture 4.6 ([BGVG12, Conjectures 4.9 and 4.10]). *Let k be a real field. Then the following equivalent statements hold (cf. Thm. 3.4 (ii)):*

- (i) $p(k(x)) = p(k((t))(x))$.
- (ii) $p(\ell(x)) \leq p(k(x))$ for all finite field extensions ℓ/k .

It is now clear (from Thm. 4.2) that the above conjecture is equivalent to the equality $p(k((t_1, t_2))) = p(k(t))$.

We now turn to the u -invariant.

Lemma 4.7. *Let k be a field. Consider the rings $B := k[[t]][x] \subseteq A := k[x][[t]]$.*

Then every $f \in A$ admits a factorization $f = u \cdot g$, where u is a unit in A and $g \in B$.

Proof. We may assume $t \nmid f$ in A . Write $f = \sum_{i \geq 0} f_i(x)t^i$ with $f_i \in k[x]$. It suffices to construct inductively two sequences $\{g_j\}_{j \geq 0}$, $\{u_j\}_{j \geq 0}$ of elements in $k[x]$ with the following two properties:

- (1) $u_0 = 1$, $g_0 = f_0$ and $\deg g_j \leq \deg f_0$ for all $j \geq 0$; and
- (2) $f_N = \sum_{j=0}^N g_j u_{N-j}$ for all $N \geq 0$.

Since $f_0(x) \neq 0$, this can be done using the Euclidean division algorithm as in our proof of Lemma 4.1. \square

Proposition 4.8 (Becher). *Let k be a field, $F = k((x, t))$, $K = \text{Frac}(k[x][[t]])$ and $L = k((t))(x)$.*

Then the maps between Witt groups $W(L) \rightarrow W(K) \rightarrow W(F)$ induced by the natural inclusions $L \subseteq K \subseteq F$ induce surjections

$$W(L)_{\text{tors}} \twoheadrightarrow W(K)_{\text{tors}} \twoheadrightarrow W(F)_{\text{tors}},$$

where the subscript “tors” means the torsion part of the group.

Proof. Here we prove the first surjection $W(L)_{\text{tors}} \twoheadrightarrow W(K)_{\text{tors}}$. The second one follows from the surjectivity of the map $W(L)_{\text{tors}} \rightarrow W(F)_{\text{tors}}$, which can be proved similarly and will be generalized in Prop. 5.4.

So consider a torsion form $\phi = \langle f_1, \dots, f_r \rangle$ over K where all the coefficients f_i lie in $A = k[x][[t]]$.

First assume k is a nonreal field. By Lemma 4.7, for each i there is a factorization $f_i = u_i g_i$, where u_i is a unit in A and $g_i \in B = k[[t]][x]$. Putting

$$\lambda_i := u_i(0, 0) \in k^* \quad \text{and} \quad h_i := \lambda_i g_i,$$

the form $\psi := \langle h_1, \dots, h_r \rangle$, defined over $L = \text{Frac}(B)$, is isomorphic to ϕ over K .

Now assume k is real. Then the form ϕ is Witt equivalent to $\psi_1 \perp \dots \perp \psi_m$ for some binary torsion forms $\psi_j = c_j \cdot \langle 1, -d_j \rangle$ with $c_j, d_j \in A$ (cf. [Pfi66, Satz. 22]). Since the form $\psi_j = c_j \cdot \langle 1, -d_j \rangle$ is torsion over K , d_j is a sum of squares in K , and we may assume it is already a sum of squares in A . By Lemma 4.1, d_j divided by a suitable square in A becomes a sum of squares in B , which we denote by d'_j . Moreover, Lemma 4.7 implies that each c_j is the product of a square in A with some element $c'_j \in B$. Thus, the form

$$\psi := c'_1 \cdot \langle 1, -d'_1 \rangle \perp \dots \perp c'_m \langle 1, -d'_m \rangle$$

is a torsion form over $L = \text{Frac}(B)$ such that $[\psi] = [\phi]$ in $W(K)$. This completes the proof. \square

Theorem 4.9. *For any field k , one has*

$$u(k((x, t))) = u(\text{Frac}(k[x][[t]])) = u(k((t))(x)) = 2 \cdot \sup\{u(\ell(x)) \mid \ell/k \text{ a finite extension}\}.$$

Proof. The last equality is proved by Becher–Grimm–Van Geel (cf. Thm. 3.4 (iii)) and is recorded here to complete the information.

Let us write $F = k((x, t))$, $K = \text{Frac}(k[x][[t]])$ and $L = k((t))(x)$. By Prop. 4.8, every torsion form ϕ over F is Witt equivalent to the base extension of a torsion form ψ over K . Hence, the anisotropic part of ϕ has dimension less than or equal to the dimension of the anisotropic part of ψ . From this it follows that $u(F) \leq u(K)$. Similarly, we get $u(K) \leq u(L)$. Finally, we have

$$u(L) = 2 \cdot \sup\{u(\ell(x)) \mid \ell/k \text{ a finite extension}\} \leq u(F)$$

by Lemma 3.6. \square

Remark 4.10. (1) If k is a nonreal field, the inequality $u(k((x, t))) \leq u(k((t))(x))$ is implicitly contained in [CDLR82]. However, even in the nonreal case the first two equalities in Thm. 4.9 seem to have escaped earlier notice.

(2) If $\hat{u}(k)$ denotes the **strong u -invariant** of k as defined in [BGVG12, §5] or [Sch09, Definition 2], i.e.,

$$\hat{u}(k) = \frac{1}{2} \sup\{u(L) \mid L \text{ an algebraic function field in one variable over } k\},$$

then Thm. 4.9 implies that $u(k((x, t))) \leq 4\hat{u}(k)$. In the nonreal case, [HHK11, Coro. 4.2] gives a generalization of this inequality.

Corollary 4.11. *If k is an algebraic function field in $d \geq 0$ variables over a real closed field k_0 , then one has*

$$u(k((x, t))) = u(\text{Frac}(k[x][[t]])) \begin{cases} = 4 & \text{if } d = 0 \\ \in [2^{d+2}, 2^{d+4} - 4d - 16] & \text{if } d \geq 1 \end{cases}$$

In particular,

$$u(\mathbb{R}(y)((x, t))) = u(\text{Frac}(\mathbb{R}(y)[x][[t]])) \in \{8, 10, 12\}.$$

Proof. If L is an algebraic function field in $d + 1 \geq 1$ variables over k_0 , we have

$$u(L) \begin{cases} = 2 & \text{if } d = 0 \\ \in [2^{d+1}, 2^{d+3} - 2d - 8] & \text{if } d \geq 1 \end{cases}$$

Here, the upper bound is due to Elman–Lam if $d \leq 1$ (cf. [EL73, Thm. 4.11]) and follows from Becher’s result ([Bec10]) if $d \geq 2$. \square

For the field k in Coro. 4.11, Pfister [Pfi82] has conjectured that $u(k) = 2^d$ holds in general. If this conjecture is true, we will get

$$u(k((x, t))) = u(\text{Frac}(k[x][[t]])) = 2^{d+2}$$

from Thm. 4.9.

5 Some general observations

In this section, we generalize some results in the previous section to the case of Laurent series in more than two variables.

Recall that a power series $f \in R_n = k[[t_1, \dots, t_n]]$ is said to be **regular in t_n** if $f(0, \dots, 0, t_n) \neq 0$ in $k[[t_n]]$. For a one-variable power series $f \in k[[t]]$, the **order** of f is its t -adic valuation, i.e., the largest integer $r \geq 0$ such that $t^r \mid f$ in $k[[t]]$.

We shall use the following Weierstrass-type theorem about power series in several variables.

Theorem 5.1. Let $f \in R_n$ be a power series that is regular in t_n and let $s \geq 0$ be the order of the power series $f(0, \dots, 0, t_n)$.

(i) There is a unique expression $f = u.P$ with u a unit in R_n and $P \in R_{n-1}[t_n]$ such that

$$\deg_{t_n} P = s \quad \text{and} \quad P(0, \dots, 0, t_n) = t_n^s.$$

(ii) For any $g \in R_n$, there is a unique expression

$$g = q.f + r$$

with $q \in R_n$ and $r \in R_{n-1}[t_n]$ such that $\deg_{t_n} r < s$. (Here $r = 0$ if $s = 0$.)

(iii) For any finite set of nonzero power series $f_1, \dots, f_m \in R_n$, there is an automorphism σ of the ring R_n such that the power series $\sigma(f_i)$, $1 \leq i \leq m$ are all regular in t_n .

Proof. (i) [ZS75, p.145, Coro. 1]. (ii) [ZS75, p.139, Thm. 5]. (iii) [ZS75, p.147, Corollary]. \square

Lemma 5.2. Let $f \in R_n$ be a power series regular in t_n and let $m \geq 1$ be an integer such that $m \leq s(k)$.

If f is a sum of m squares in R_n , then there exist elements $a \in R_n$ and $b \in R_{n-1}[t_n]$ with b a sum of m squares in $R_{n-1}[t_n]$ such that $f = a^2 b$.

Proof. The proof is similar to that of Lemma 4.1.

Write $f = \sum_{i=1}^m g_i^2$ with $g_i \in R_n$. Let s be the order of $f(0, \dots, 0, t_n) \in k[[t_n]]$. If $s = 0$, then f is a unit in R_n and $\alpha := f(0, \dots, 0) \in k^*$. In this case $\alpha^{-1}f$ is a square in R_n and

$$\alpha = f(0, \dots, 0) = \sum_{i=1}^m g_i(0, \dots, 0)^2 \in D_k(m) \subseteq D_{R_{n-1}[t_n]}(m).$$

So we may take $a \in R_n$ such that $a^2 = \alpha^{-1}f$ and $b = \alpha$.

Assume next $s > 0$, so that $0 = \sum_{i=1}^m g_i(0, \dots, 0)^2$ in k . Since $m \leq s(k)$ by assumption, one has $g_i(0, \dots, 0) = 0$ for every i . By Thm. 5.1 (ii), for each g_i , one can write $g_i = h_i f + r_i$, where $h_i \in R_n$ and $r_i \in R_{n-1}[t_n]$ with $\deg_{t_n} r_i < s$. Then the power series

$$u := 1 - 2 \sum_{i=1}^m g_i h_i + f \sum_{i=1}^m h_i^2,$$

is a unit and is a square in R_n since $u(0, \dots, 0) = 1$, and one has

$$\sum r_i^2 = \sum (g_i - h_i f)^2 = \sum g_i^2 - 2f \sum g_i h_i + f^2 \sum h_i^2 = f u.$$

Now taking $a \in R_n$ such that $a^2 = u^{-1}$ and $b = \sum r_i^2$ finishes the proof. \square

Proposition 5.3. Let k be a real field and $m \geq 1$ an integer.

Then given finitely many elements $f_1, \dots, f_r \in D_{R_n}(m)$, there is an automorphism σ of the ring R_n such that $\sigma(f_i) \in R_n^2 \cdot D_{R_{n-1}[t_n]}(m)$ for every $1 \leq i \leq r$. If $n \leq 2$, we may take σ to be the identity.

Proof. The case with $n = 1$ is left to the reader. If $n = 2$, this is part of Lemma 4.1. For general n , it suffices to apply Thm. 5.1 (iii) to get an automorphism σ such that all the $\sigma(f_i)$ are regular in t_n . Then the result follows from Lemma 5.2. \square

In general, the automorphism σ in the above proposition may not preserve the subring $R_{n-1}[t_n]$.

Proposition 5.4. *For every torsion form ϕ over F_n , there is an automorphism σ of F_n and a torsion form ψ over $F_{n-1}(t_n)$ such that $\sigma_*[\phi] = [\psi] \in W(F_n)$, where*

$$\sigma_* : W(F_n) \longrightarrow W(F_n); \quad \langle f_1, \dots, f_r \rangle \longmapsto \langle \sigma(f_1), \dots, \sigma(f_r) \rangle$$

denotes the automorphism of the Witt group $W(F_n)$ induced by σ . If $n \leq 2$, one can take σ to be the identity.

Proof. We may assume $\phi = \langle f_1, \dots, f_r \rangle$, where all the coefficients f_i lie in R_n .

First assume k is a nonreal field. By Thm. 5.1 (i) and (iii), there is an automorphism σ of R_n such that the power series $\sigma(f_i)$ admits a factorization $\sigma(f_i) = u_i g_i$, where u_i is a unit in R_n and $g_i \in R_{n-1}[t_n]$. When $n \leq 2$, one can take σ to be the identity. (If $n = 2$, one has $f_i = t_1^{r_i} f'_i$ for some $r_i \geq 0$ and f'_i regular in t_2 .) Putting

$$\lambda_i := u_i(0, \dots, 0) \in k^* \quad \text{and} \quad h_i := \lambda_i g_i,$$

we get a form $\psi := \langle h_1, \dots, h_r \rangle$ which is defined over $F_{n-1}(t_n)$ and isomorphic to

$$\sigma(\phi) := \langle \sigma(f_1), \dots, \sigma(f_r) \rangle.$$

over F_n .

Now consider the case with k real. Then the form ϕ is Witt equivalent to $\psi_1 \perp \dots \perp \psi_m$ for some binary torsion forms $\psi_j = c_j \cdot \langle 1, -d_j \rangle$ with $c_j, d_j \in R_n$ (cf. [Pfi66, Satz. 22]). Since the form $\psi_j = c_j \cdot \langle 1, -d_j \rangle$ is torsion over F_n , d_j is a sum of squares in F_n , and we may assume d_j is already a sum of squares in R_n .

By Prop. 5.3, there is an automorphism σ of R_n , which we may take to be the identity if $n \leq 2$, such that each of the $\sigma(d_j)$ is a sum of squares in $R_{n-1}[t_n]$ up to a square in R_n . In fact, we may further assume that σ is chosen such that each $\sigma(c_j)$ admits a factorization $\sigma(c_j) = u_j e_j$, where u_j is a unit in R_n and $e_j \in R_{n-1}[t_n]$. (For $\sigma \geq 3$, it suffices to choose σ such that all the $\sigma(d_j)$ and $\sigma(c_j)$ are regular in t_n .) Thus, there are elements $d'_j, c'_j \in R_{n-1}[t_n]$ such that d'_j is a sum of squares in $R_{n-1}[t_n]$ and

$$d'_j \sigma(d_j)^{-1} \in R_n^2, \quad c'_j \sigma(c_j)^{-1} \in R_n^2$$

for every j . Now the form

$$\psi := c'_1 \cdot \langle 1, -d'_1 \rangle \perp \dots \perp c'_m \langle 1, -d'_m \rangle$$

is a torsion form over $F_{n-1}(t_n)$ with the desired property. \square

Corollary 5.5. *For any field k and any $n \geq 1$, one has $p(F_n) \leq p(F_{n-1}(t))$ and $u(F_n) \leq u(F_{n-1}(t))$.*

Proof. First consider the assertion about the Pythagoras number. If k is nonreal, then $p(F_n) = s(k) + 1 = p(F_{n-1}(t))$ by Prop. 3.3. So let us assume k is real and $m := p(F_{n-1}(t)) < \infty$. We want to show that every sum of squares f in F_n is a sum of m squares. Let σ be an automorphism of F_n as in Prop. 5.3. Then we have $\sigma(f) \in D_{F_n}(m)$ and hence $f \in D_{F_n}(m)$.

For the assertion about the u -invariant, let $u = u(F_{n-1}(t)) \leq \infty$ and let ϕ be a torsion form over F_n . We need to show that the anisotropic part ϕ_{an} of ϕ has dimension $\dim \phi_{an} \leq u$. By Prop. 5.4, there is an automorphism σ of the field F_n such that $[\sigma(\phi)] = \sigma_*[\phi] = [\psi] \in W(F_n)$ for some torsion form ψ over $F_{n-1}(t_n)$. Hence,

$$\dim \phi_{an} = \dim \sigma(\phi)_{an} \leq \dim \psi_{an} \leq u = u(F_{n-1}(t_n)),$$

completing the proof. \square

6 The Pythagoras number of Laurent series fields in three variables

Theorem 6.1. *Let k be a real field such that $p(k(x, y)) \leq 4$ (e.g., k real closed). Then*

$$p(k((t_1, t_2, t_3))) = p(k((t_1, t_2))(t)) = 4.$$

Proof. In fact, one has $p(K(x, y)) \geq 4$ for any real (but not necessarily real closed) field K . (The classical proof using the Motzkin polynomial [Pfi95, p.96] can be easily extended as explained in [Gri12, p.2].) So the hypothesis $p(k(x, y)) \leq 4$ is equivalent to $p(k(x, y)) = 4$.

We have thus $p(F_3) \geq p(k(x, y)) = 4$ by Lemma 3.5 (ii). (As pointed out by Becher, one can also prove $p(F_3) \geq 4$ by using the Motzkin polynomial.) In view of Coro. 5.5, we need only to show $p(F_2(t)) \leq 4$. By a theorem of Pfister (cf. [Lam05, p.397, Examples XI.5.9 (3)]), this is equivalent to saying that $s(L) \leq 2$ for every finite nonreal extension L of F_2 .

Fix such an extension L/F_2 and let R be the integral closure of $k[[t_1, t_2]]$ in L . Call a discrete valuation w of L **divisorial** if there is a regular integer scheme X equipped with a proper birational morphism $X \rightarrow \text{Spec}(R)$ such that w is defined by a codimension 1 point of X . It is proved in [Hu10, Thm. 1.1] that the isotropy of quadratic forms of rank 3 or 4 over L satisfies the local-global principle with respect to all divisorial valuations of L .

By this local-global principle, it suffices to show that for every *divisorial* discrete valuation w on L , the completion L_w has level $s(L_w) \leq 2$. For such a discrete valuation, L_w is nonreal as L is, and the residue field $\kappa(w)$ is (isomorphic to) either a finite *nonreal* extension of $k(t)$ or the fraction field of a complete discrete valuation ring whose residue field ℓ is a finite nonreal extension of k . In the former case, the hypothesis on $k(x, y) = k(x)(y)$ implies that $s(\kappa(w)) \leq 2$ (by Pfister's theorem). In the latter case, we have $s(\kappa(w)) = s(\ell) \leq 2$ since $p(k(t)) \leq p(k(x, y)) \leq 4$. Hence in any case, we get $s(L_w) = s(\kappa(w)) \leq 2$ as desired. \square

It remains an open problem whether $p(\mathbb{R}((t_1, \dots, t_n))) < \infty$ when $n \geq 4$.

Remark 6.2. (1) The hypothesis $p(k(x, y)) \leq 4$ in Thm. 6.1 is satisfied if k is a hereditarily euclidean field (cf. [BVG09, Coro. 4.6]).

(2) In the situation of Thm. 6.1, one has $2 \leq p(L) \leq 3$ for every finite extension L of $F_2 = k((t_1, t_2))$ by [BVG09, Thm. 3.5].

7 Two conjectures in the general case

(7.1) Let k be a field and $F_n = k((t_1, \dots, t_n))$ a Laurent series field in n variables over k . For any $n \geq 2$, we have shown (cf. Lemma 3.5 and Coro. 5.5)

$$p(F_{n-1}(t)) \geq p(F_n) \geq \sup\{p(\ell(t_1, \dots, t_{n-1})) \mid \ell/k \text{ a finite field extension}\}.$$

Now consider the following statements for every $n \geq 2$:

$\mathcal{P}_n(k)$: $p(F_n) = \sup\{p(\ell(t_1, \dots, t_{n-1})) \mid \ell/k \text{ a finite field extension}\}.$

$\mathcal{P}'_n(k)$: $p(F_n) = p(k(t_1, \dots, t_{n-1}))$.

$\mathcal{F}_n(k)$: $p(F_{n-1}(t)) = \sup\{p(\ell(t_1, \dots, t_{n-1})) \mid \ell/k \text{ a finite field extension}\}.$

$\mathcal{F}'_n(k)$: $p(F_{n-1}(t)) = p(k(t_1, \dots, t_{n-1}))$.

Prop. 3.3 implies that all these statements hold if k is nonreal. In the real case, Becher, Grimm and Van Geel (cf. Thm. 3.4 (ii)) have proved $\mathcal{F}_2(k)$, and Thm. 4.7 confirms that $\mathcal{P}_2(k)$ is always true. Also, we have shown (in Thm. 6.1) that $\mathcal{P}'_3(k)$ and $\mathcal{F}'_3(k)$ hold for real fields k with $p(k(x, y)) \leq 4$. If the conjecture of Becher, Grimm and Van Geel (Conjecture 4.6) is true for all real fields, then $\mathcal{P}'_n(k)$ (resp. $\mathcal{F}'_n(k)$) is equivalent to $\mathcal{P}_n(k)$ (resp. $\mathcal{F}_n(k)$).

In general, we conjecture that $\mathcal{P}_n(k)$ and $\mathcal{F}_n(k)$ hold for all fields k and all $n \geq 2$.

Conjecture 7.2. *For every integer $n \geq 2$ and every field k , one has*

$$p(F_{n-1}(t)) = p(F_n) = \sup\{p(\ell(t_1, \dots, t_{n-1})) \mid \ell/k \text{ a finite field extension}\}.$$

As in Example 4.3, if the above conjecture is true, we will get finiteness results on $p(k((t_1, \dots, t_n)))$ and $p(k((t_1, \dots, t_{n-1}))(t))$ in a number of special cases, e.g., $k = \mathbb{R}$ or $k = \mathbb{Q}$.

For the u -invariant, similar considerations lead us to propose the following conjecture.

Conjecture 7.3. *For every integer $n \geq 2$ and every field k , one has*

$$u(F_{n-1}(t)) = u(F_n) = 2 \sup\{u(\ell(t_1, \dots, t_{n-1})) \mid \ell/k \text{ a finite field extension}\}.$$

For a real closed field k and $n \geq 2$, if the equality $u(k(t_1, \dots, t_{n-1})) = 2^{n-1}$ holds as conjectured by Pfister [Pfi82], then the above conjecture implies $u(k((t_1, \dots, t_n))) = 2^n$.

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